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A Method of Approximating Stiffness Matrices for Some Pseudodifferential Equations

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Most of the entries of stiffness matrices that arise in spline and FE solution of pseudodifferential equations do not include singular integrals, while a few do. The former can be approximated with a variety of techniques including numerical integration and asymptotic formulas. In the present paper we show how to avoid evaluating singular integrals once nonsingular entries have been computed. This approach works in conjunction with Galerkin or least squares uniform spline or FE approximation in solving homogeneous pseudodifferential equations.

1. INTRODUCTION

The boundary integral method (BIM) often emerges as a method of choice for solving various physical problems [5, 9, 12,.14]. Advantages offered by boundary integral formulations are well known:

—reduction by 1 of the dimension of equations:

—representation of an approximate solution in analytic form.

The main drawback is in that the resulting integral equation is considerably more complicated and hard to handle numerically than the original differential equation.

A customary engineering approach to solving boundary integral equations utilizes collocation paired with piecewise constant approximation of the unknown solution [5]. In this setting the approach is known as the boundary element method (BEM). In recent years an alternative to BEM in the form of the Galerkin method has been rigorously investigated [2, 4, 6, 11, 14, 15]. Allowing for more sophisticated kinds of approximation, the Galerkin method is theoretically superior to BEM. Also it is equally applicable to solving unilateral problems [1, 2, 6]. However, in practice, the Galerkin method leads to very involved computations of stiffness matrices.

It is our purpose in the present note to discuss a simple, fast, and accurate method for computing these stiffness matrices. It is best to consider the general framework of pseudodifferential equations. For different kinds of pseudodifferential

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equations that appear in practice, see [6, 14]. In Section 3 we shall concentrate on two important problems from the elasticity theory.

The method for computing stiffness matrices as presented in this note is not universal and requires the following two conditions:

—the operator of the governing equation is homogeneous with constant coefficients, see (2.3) and [6, 14];

—the grid is uniform, see (2.4).

Neither condition imposes serious restrictions on applicability of the method. As far as BIM is concerned most operators would satisfy the first requirement, while the second, on the other hand, is standard for the Galerkin solution of integral equations on flat surfaces and should be satisfied only locally on curved surfaces.

Under the above two conditions, entries of the stiffness matrices depend (see Section 2) on the meshsize parameter h through a multiplicative constant and, in fact, are to be computed just once. However, the complexity of the integrals involved is at times such that in some instances [7, 10] the integrals are approximated to as few as 5 significant digits only.

The method we are about to describe is based on the following observation. Of the entries of stiffness matrices only a few include singular integrals. The majority can be split into sums of integrals of analytic functions. The latter can, in principal, be evaluated to an arbitrary degree of precision. Roughly speaking, in the circumstances described above it is possible to form an algebraic system of linear equations with singular entries as its unknowns and the right-hand side related to the analytic entries. This allows one to approximate singular entries as accurately (and with little additional expense) as the analytic ones.

The method can be used to approximate principal parts of entries in the framework of Galerkin–collocation method on curved surfaces, see [8, 14, 15].

It is next to impossible to compare computational efforts required for evaluating the stiffness matrices. And because the entries must be computed once and for all, such comparison does not bear any significance. What is more important is the amount of preparatory work and effort spent on implementation of a particular strategy for computing those integrals. None of the methods used so far [1, 7, 8, 10, 15], of which all are based on some kind of numerical integration, is sufficiently general to work without major modifications for a different kind of equations than the one it was devised for.

2. Uniform Finite Dimensional Spaces

We consider a problem of solving

$$Au = f \tag{2.1}$$

in a bounded open region $D \subset \mathbb{R}^N$, $N \geqslant 1$, with A a pseudodifferential operator

$$Av = F^{-1}(A(\xi)\, Fv), \qquad \xi \in \mathbb{R}^N, \tag{2.2}$$

where F and F^{-1} stand for the (generalized) Fourier and inverse Fourier transformations, respectively. $A(\xi)$, which is the symbol of the operator A, is assumed to be homogeneous of order α, i.e.,

$$A(t\xi) = t^{\alpha}A(\xi), \qquad t > 0. \tag{2.3}$$

Numerous examples of such operators can be found in [6, Section 3].

We assume there is given a fuction $\psi \in L^2(\mathbb{R}^N)$ with compact support, and for all $h > 0$ define a family of functions

$$\psi_{p,h}(x) = \psi\left(\frac{x}{h} - p\right), \tag{2.4}$$

$p \in \mathbb{N}^N$. Let, e.g., $N_h = \{p : ph \in D\}$ and $V_h^D = \operatorname{span}\{\psi_{p,h} : p \in N_h\}$: then the Galerkin method consists in finding $u_h \in V_h^D$ such that

$$\langle Au_h, v_h \rangle = \langle f, v_h \rangle, \qquad \text{all} \quad v_h \in V_h^D, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^N)$ extended to the pairing of some spaces of generalized functions.

If $u_h = \sum_{p \in N_h} c_p \psi_{p,h}$ then (2.5) is equivalent to

$$\sum_{p \in N_h} c_p \langle A\psi_{p,h}, \psi_{q,h} \rangle = \langle f, \psi_{q,h} \rangle, \qquad q \in N_h. \tag{2.6}$$

It is the question of evaluating the stiffness matrix $(\langle A\psi_{p,h}, \psi_{q,h} \rangle)$ that we intend address in the present note. In the following lemma, which can be easily shown by a few applications of the Fourier transform, we state an essential property of the entries $\langle A\psi_{p,h}, \psi_{q,h} \rangle$.

LEMMA 2.1. *Assume A satisfies* (2.3). *Then*

$$\langle A\psi_{p,h}, \psi_{q,h} \rangle = h^{N-\alpha} \langle A\psi_{0,1}, \psi_{p-q,1} \rangle.$$

For simplicity we shall write $\psi_p = \psi_{p,1}$. Thus we have

$$\langle A\psi_{p,h}, \psi_{q,h} \rangle = h^{N-\alpha} \langle A\psi_0, \psi_{p-q} \rangle \tag{2.7}$$

which means that once the entries $\langle A\psi_0, \psi_p \rangle$ have been computed we may proceed with the Galerkin method (2.6) for any $h > 0$. Therefore our problem reduces to evaluating $\langle A\psi_0, \psi_p \rangle$.

At this point we need an additional assumption regarding the spaces V_h^D. We assume that

$$V_1 \subset V_{1/2} \qquad (\text{or} \quad \psi \in V_{1/2}), \tag{2.8}$$

where $V_h = \operatorname{span}\{\psi_{p,h} : p \in \mathbb{N}^N\}$.

This assumption holds for all common finite element spaces and spline spaces (and their tensor products) over uniform grids. For finite element spaces it follows by construction, while for spline spaces (2.8) it is a consequence of the following lemma.

LEMMA 2.2. *Let* $\{B_{i,n}(u)\}_i$ *be the family of uniform B-splines of degree* n *with* $(n-1)$ *continuous derivatives. Then*

$$B_{0,n}\left(\frac{u}{2}\right)=\frac{1}{2^n}\sum_{j=0}^{n+1} C_j^{n+1} B_{j,n}(u). \tag{2.9}$$

We believe the identity (2.9) is known but have failed to find a proper reference. It is quite easily shown by comparing Fourier transforms of both sides. From (2.8),

$$\psi=\sum_q k_q \psi_{q,1/2},$$

where the sum is finite. Hence from (2.4),

$$\psi_p=\sum_q k_q \psi_{2p+q,1/2}. \tag{2.10}$$

Now introduce $a_p=\langle A\psi_0, \psi_p\rangle$ and substitute the sum (2.10) into a_p:

$$\begin{aligned} a_p &= \left\langle A\left[\sum_q k_q \psi_{q,1/2}\right], \sum_r k_r \psi_{2p+r,1/2}\right\rangle \\ &= \sum_{q,r} \langle A\psi_{q,1/2}, \psi_{2p+r,1/2}\rangle k_q k_r. \end{aligned}$$

By Lemma 2.1 one then has

$$a_p=\sum_{q,r} (k_q k_r 2^{\alpha-N})\, a_{2p+r-q}. \tag{2.11}$$

Thus every coefficient a_p is a linear combination of some other coefficients.

Note here that if $\operatorname{supp}\psi_0 \cap \operatorname{supp}\psi_p \neq \varnothing$ then a_p includes, in general, a singular integral, while if $\operatorname{supp}\psi_0 \cap \operatorname{supp}\psi_p = \varnothing$, a_p can be split into a sum of integrals of analytic functions. Assuming that the job of computing the latter is much simpler than in the former case, we want to use (2.11) to express "difficult" a_p's in terms of the "easy" ones. In general, (2.11) leads to an algebraic system of a few linear equations.

Remark 2.3. One may use various criteria to differentiate between difficult and easy terms a_p. On the whole, a method would more accurately compute a_p's with larger p's than with smaller ones.

In the next section we use (2.11) to approximate the stiffness matrix $\{a_p\}$ for two problems that arise in the elasticity theory.

3. Crack and Stamp Problems

In this section we consider only $N=2$ and $A(\xi)=|\xi|^\alpha$, $|\alpha|<2$, where $\alpha\in\mathbb{R}$, $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, $|\xi|=(\xi_1^2+\xi_2^2)^{1/2}$. Written explicitly, (2.2) becomes

$$(Av)(x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\xi|^\alpha\,\tilde{v}(\xi)\,e^{-ix\cdot\xi}\,d\xi \tag{3.1}$$

with $\tilde{v}=Fv$, $x=(x_1,x_2)\in\mathbb{R}^2$, and $x\cdot\xi=x_1\xi_1+x_2\xi_2$. As is well known, $\alpha=1$ and $\alpha=-1$ correspond to the Crack and Stamp problems, respectively, see [4, 6, 9, 11, 13].

To solve (2.1) we choose bilinear finite elements formed as in (2.4) with

$$\psi(x)=\begin{cases}(1-|x_1|)(1-|x_2|), & |x_1|,|x_2|\leqslant 1\\ 0, & \text{otherwise.}\end{cases} \tag{3.2}$$

The spaces V_h can be thought of as the tensor products of one-dimensional linear spline spaces. Therefore as an application of Lemma 2.2 we have

Lemma 3.1. *With $\psi_{p,h}$ defined by* (2.4) *and* (3.2) *the following identity holds*:

$$\begin{aligned}\psi_{p,h}(x)=\psi_{2p,h/2}(x)&+\tfrac{1}{2}(\psi_{2p_1+1,2p_2,h/2}(x)+\psi_{2p_1-1,2p_2,h/2}(x)\\&+\psi_{2p_1,2p_2+1,h/2}(x)+\psi_{2p_1,2p_1-1,h/2}(x))\\&+\tfrac{1}{4}(\psi_{2p_1+1,2p_2+1,h/2}(x)+\psi_{2p_1-1,2p_2+1,h/2}(x)\\&+\psi_{2p_1+1,2p_2-1,h/2}(x)+\psi_{2p_1-1,2p_2-1,h/2}(x)),\end{aligned} \tag{3.3}$$

where $p=(p_1,p_2)\in\mathbb{N}^2$.

Since $A(\xi)=|\xi|^\alpha$, (2.7) becomes

$$\langle A\psi_{p,h},\psi_{q,h}\rangle=\frac{h^{2-\alpha}}{(2\pi)^2}\int_{\mathbb{R}^2}|\xi|^\alpha\,\tilde{\psi}^2(\xi)\,e^{i(p-q)\cdot\xi}\,d\xi. \tag{3.4}$$

In other words, a_{pq} actually depends symmetrically on $|p_1-q_1|$ and $|p_2-q_2|$. Therefore it is sufficient to evaluate a_p for $0\leqslant p_2\leqslant p_1$. To set up the algorithm we arrange those p's in the lexicographic order

$$(0,0),(1,0),(1,1),(2,0),\ldots. \tag{3.5}$$

Now, choose some $0<K\in\mathbb{N}$ and consider a_p's with $p_1\leqslant K$. For each such p, split the sum in (2.11) into two parts: one with $\max\{|2p_1+r_1-q_1|, |2p_2+r_2-q_2|\}>K$, and the order for which the inverse inequality holds. As we shall see presently the first part is approximated quite accurately and easily for K sufficiently, but not unreasonably large.

The vector a^K consists of a_p's arranged according to (3.5) with $p_1 \leqslant K$. Then $a^K \in \mathbb{R}^{(K+1)(K+2)/2}$. The procedure leads to a linear system

$$a^K = B^K a^K + b^K \tag{3.6}$$

of $(K+1)(K+2)/2$ equations with as many unknowns. Matrix B^K is formed by sums of products of 1, $\frac{1}{2}$, $\frac{1}{4}$ (see (3.3)), and components of the vector b^K are linear combinations of a_p's with $p > K$.

PROPOSITION 3.2. *For $K > 1$ and $\alpha \neq 0$, system (3.6) has a unique solution.*

Proof. First of all, it is clear that (3.6) has a solution since the actual entries a_p do satisfy the system. We use induction on K to prove the uniqueness. For $K = 2$, we have a 6×6 system and by direct inspection find that the matrix B^2 has the form

$$2^\alpha \begin{bmatrix} 9\cdot 2^{-4} & 3\cdot 2^{-1} & 1 & * & * & * \\ 3\cdot 2^{-5} & 2^{-1} & 2^{-1} & * & * & * \\ 2^{-6} & 2^{-3} & 2^{-2} & * & * & * \\ 0 & 0 & 0 & 3\cdot 2^{-5} & 2^{-3} & * \\ 0 & 0 & 0 & 2^{-6} & 2^{-4} & * \\ 0 & 0 & 0 & 0 & 0 & 2^{-6} \end{bmatrix},$$

where stars stand for some nonzero numbers. Its eigenvalues are easily found to be

$$2^\alpha, 2^{\alpha-2}, 2^{\alpha-3}, 2^{\alpha-4}, 2^{\alpha-5}, 2^{\alpha-6}. \tag{3.7}$$

Therefore if, as we have assumed, $|\alpha| < 2$ and $\alpha \neq 0$ the system (3.6) has a unique solution for $K = 2$.

Next, assume it has a unique solution for $K = m$, $m > 1$. Consider (3.6) with $K = m+1$. We claim that in rows of B^{m+1} corresponding to p's with $p_1 = m+1$, all the entries are 0. Indeed, from (3.3) and $a_p = \langle A\psi_0, \psi_p \rangle$ it follows that in the rows of B^{m+1} all entries are 0 provided

$$2p_1 - 2 > m + 1.$$

With $p_1 = m+1$ this reduces to

$$2m > m + 1$$

or $m > 1$. Hence a_p's with $p_1 = m+1$ coincide with the corresponding components of b^{m+1} and the latter are computed in a well-defined manner.

Finally, if we transfer all a_p's, $p_1 = m+1$, from $B^{m+1}a^{m+1}$ to b^{m+1} and drop all redundant equations we shall have the system $a^m = B^m a^m + b^m$ which by the inductive assumption does have a unique solution. This constitutes the part of $(m+1)(m+2)/2$ first components of a^{m+1}. Therefore all of a^{m+1} is uniquely determined by (3.6). ∎

From the propsition we see that it is possible to compute all entries a_p once we know a_p with $\max\{|p_1|, |p_2|\} > K$ for some $K > 1$. To evaluate the latter there is a very handy asymptotic formula, [6, Lemma 26.1].

From (3.4) it follows that a_p's are values of the Fourier transform of the function $|\xi|^\alpha \tilde{\psi}(\xi)$ at $x = p$. Since

$$\tilde{\psi}^2(\xi) = \left(\frac{\sin^2(\xi_1/2)}{(\xi_1/2)^2} \cdot \frac{\sin^2(\xi_2/2)}{(\xi_2/2)^2}\right)^2,$$

the Taylor series of $\tilde{\psi}^2(\xi)$ is readily available. In particular,

$$\tilde{\psi}^2(\xi) = 1 - \frac{|\xi|^2}{6} + \frac{|\xi|^4}{80} + \frac{\xi^2\xi_2^2}{360} - \frac{17\,|\xi|^6}{3024} - \frac{\xi_1^2\xi_2^2\,|\xi|^2}{2520} + \cdots. \tag{3.8}$$

Now (3.8) and some known properties of the Fourier transform lead to the following asymptotic formulas.

LEMMA 3.3. *As* $|p| \to \infty$ *the following expansions hold*:

$$a_p = 2\pi\left(\frac{1}{|r|} + \frac{1}{6\,|r|^3} + \frac{19}{240\,|r|^5} + \frac{7r_1^2r_2^2}{24\,|r|^9} + \frac{61}{672\,|r|^7} + \frac{3r_1^2r_2^2}{8\,|r|^{11}}\right) + O(|r|^{-9}) \tag{3.9.1}$$

if $\alpha = -1$ (*Stamp problem*), *and*

$$a_p = -\frac{1}{2\pi}\left(\frac{1}{|r|^3} + \frac{3}{2\,|r|^5} + \frac{41}{16\,|r|^7} + \frac{21r_1^2r_2^2}{8\,|r|^{11}} + \frac{499}{96\,|r|^9} + \frac{99r_1^2r_2^2}{8\,|r|^{13}}\right) + O(|r|^{-11}) \tag{3.9.2}$$

if $\alpha = -1$ (Crack problem).

4. Numerical Results

We have solved the system (3.6) for $\alpha = \pm 1$ using the standard IMSL routine LEQT1F in double precision on a PRIME 950 machine. Since (3.9) does not provide an explicit accuracy estimate we ran our program with different K. Some sample computations are summarized in the Tables I and II.

TABLE I

Standard IMSL Routine LEQT 1F for $\alpha = -1$

K \ p	(0, 0)	(0, 1)	(1, 2)	(3, 4)
8	0.132576662426*E*2	0.742731887240*E*1	0.292284568059*E*1	0.126532434331*E*1
9	0.132576662471*E*2	0.742731887498*E*1	0.292284568162*E*1	0.126532434350*E*1
10	0.132576662489*E*2	0.742731887599*E*1	0.292284568200*E*1	0.126532435354*E*1
11	0.132576662497*E*2	0.752731887649*E*1	0.292284568217*E*1	0.126532434354*E*1
12	0.132576662501*E*2	0.742731887670*E*1	0.292284568223*E*1	0.126532434354*E*1
13	0.132576662502*E*2	0.742731887677*E*1	0.292284568224*E*1	0.126532434354*E*1

TABLE II

Standard IMSL Routine LEQT 1F for $\alpha = 1$

K \ p	(0, 0)	(0, 1)	(1, 2)	(3, 4)
15	0.92500318122	$0.2826451812E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$
16	0.92500318137	$0.2826451812E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$
17	0.92500318146	$0.2826451813E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$
18	0.92500318152	$0.2826451813E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$
19	0.92500318155	$0.2826451813E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$
20	0.92500318157	$0.2826451813E-2$	$-0.2238967954E-2$	$-0.1356833019E-3$

As it was noted in the proof of Proposition 3.2, some rows of the matrix B contain only zeros. This is true for the rows corresponding to the p's that satisfy

$$2p_1 - 2 > K. \tag{4.1}$$

Taking (4.1) into account reduces the size of the system by a factor of 2–4, depending on the magnitude of K. Furthermore, using the same idea repeatedly and a kind of a block backsubstituion, the problem of solving (3.6) actually reduces to that of solving a 6×6 system with the matrix B^2 and updated column b^2. Thus the method is extremely fast.

It can also be shown that $\alpha < 0$ the iterative process based on (3.6) converges. The number of iterations needed to attain convergence within 10^{-15} is a slowly increasing function of K, which for $K = 7, ..., 13$ ranged between 40 and 60 iterations. The solution coincided with that obtained by the direct method.

Remark 4.1. The improved accuracy in computing the integrals a_p did not have any serious impact on accuracy of numerical solution to the Stamp problem. This may be explained by the fact that for the unilateral Stamp problem the convergence in L^2-norm is $O(h^{1/2})$ [4, 6]. However, an improved formulation that uses weighted finite elements (see [3]) promises convergence $O(h^2)$ in L^2-norm. Thus we hope that in this case the effect of errors in computing a_p's will be more pronounced.

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